SATELLITE DYNAMICS ABOUT ASTEROIDS: COMPUTING POINCARÉ MAPS FOR THE GENERAL CASE

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1. Introduction

The study of orbital dynamics of spacecraft about non-spherical bodies has usually been restricted to the "planetary" case where the body is close to an oblate spheroid, with only a relatively small degree of equatorial ellipticity. When investigating; spacecraft dynamics about asteroids, the situation is drastically different as the asteroid shape is usually very distended with many irregular features. Research into the dynamics of particles about asteroids accounting for their generalized shape has only been initiated relatively recently ([I], [4], [5]). This communication outlines an algorithm to compute Poincaré maps and their associated monodromy matrices about arbitrary shapes. This capability is vital for systematic investigations of motion in this problem.

2. Problem Formulation

The equations of motion for a particle about an asteroid are most convenient in an asteroid-fixed coordinate system as the asteroid gravity field may be evaluated without transformation. Conversely, if stated in an inertial frame, the potential becomes an explicit function of time.

Denote the body-fixed position vector as r and the Body-fixed velocity vector as $\dot{\mathbf{r}} = \mathbf{v}$. The Lagrangian of the particle in the body-fixed system is:

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}(\dot{\mathbf{r}} + \Omega \times \mathbf{r}) \cdot (\dot{\mathbf{r}} + \Omega \times \mathbf{r}) + U(\mathbf{r}) \qquad (1)$$

where Ω is the asteroid rotational velocity vector expressed in the body-fixed frame and U is the gravitational force potential of the asteroid. The system may be expressed in Hamiltonian form via the well-known Legendre transformation. The Hamiltonian expressed in Lagrangian variables is:

$$J = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{1}{2}(\Omega \times \mathbf{r}) \cdot (\Omega \times \mathbf{r}) - U(\mathbf{r}) \qquad (2)$$

Taking the time derivative yields:

$$\dot{J} = \dot{\Omega} \cdot [(\dot{\mathbf{r}} + \Omega \times \mathbf{r}) \times \mathbf{r}] \quad (3)$$

If the asteroid is rotating uniformly, then $|\Omega|$ is constant (and is directed along a principal moment of inertia), and the Hamiltonian is conserved. This case has analogies with the 3-dimensional restricted three-body problem, although it has none of the geometric symmetries present in that problem. Still, it is possible to define zero-velocity surfaces, compute equilibrium points and periodic orbit families and to compute stability parameters [4].

If the asteroid is not uniformly rotating, then Ω is a time-periodic function in the body-fixed frame (assuming that external torques on the asteroid are negligible). The motion of Ω follows a well defined path described by elliptic functions [2]. In this case the equations of motion are time-periodic and the Hamiltonian is not conserved. This case has analogies with the elliptic three-body problem, although it also lacks the geometric symmetry present in that problem. Analysis of this problem is more difficult than the uniformly rotating case, although the computation of Poincaré maps is simple, as explained later.

The asteroid gravitational force potentials are generally derived from shape models with an assumed value of constant density, Given an asteroid shape and density, there are two fundamental approaches to constructing the force potential. The most common way is to expand the potential in a spherical harmonic expansion and directly compute the coefficients of the gravity field by evaluating integrals over the surface of the asteroid. Such an expansion is always truncated at some order, and thus is never a true representation. Additionally, the series expansion does not converge when inside the smallest sphere circumscribed about the asteroid and thus is not useful for studying the gravitational field close to the body.

A second way to model the potential is to discretize the asteroid shape into N tetrahedra. Then, using the closed-form solutions for the potential of a tetrahedron [6], the potential is found by summing over all the tetrahedra. This approach is more time intensive, yet is an exact result for a given asteroid discretization. Moreover, as this potential satisfies Laplace's equation external to the body and Poisson's equation internal to the body, it is an excellent candidate for evaluating motion close to or on the asteroid surface.

3. Numerical Computation of Poincaré Maps

To compute and continue families of periodic orbits and to ascertain their stability it is necessary to compute Poincaré maps and their associated monodromy matrices [3]. Due to the generic irregularity of asteroid shapes there are no geometrical symmetries with which one may reduce or simplify the dynamical problem. Also, the construction of canonical coordinates close to action-angle variables is, in general, impossible due to the large deviation of the system from integrable cases. Thus one must develop numerical approaches to take the place of analytic reductions.

If the central body is undergoing non-uniform rotation, then the computation of Poincaré maps is simple, as the period of the body-fixed rotational velocity vector is used as the Poincaré surface, reducing the 4DOF Hamiltonian system to a 3DOF Hamiltonian mapping. Then in the vicinity of periodic, orbits the full 6 x 6 state transition matrix of the Lagrangian system will generically not have unity eigenvalues and can be used to converge upon fixed points in the 6 dimensional mapping.

in the case of a uniformly rotating asteroid the choice of a Poincaré surface is not as obvious, and in the neighborhood of periodic orbits the 6 x 6 state transition matrix has two unity eigenvalues. In this situation a 4-dimensional Poincaré map must be defined and its attendant 4 x 4 monodromy matrix computed. In the following discussion a numerical procedure to do this is outlined in terms of Lagrangian variables.

3.1. FO1LMULATION

Assume a Lagrangian description in Cartesian coordinates as this is the simplest system to work with from a numerical point of view.

$$\mathbf{r}_i = r_i(t, \mathbf{r}_o, \mathbf{v}_o),\ \mathbf{v}_i = v_i(t,\ \mathbf{r}_o, \mathbf{v}_o);\ i = 1,2,3 \tag{4}$$

Also assume that the asteroid is uniformly rotating, and thus the Hamiltonian $J(\mathbf{r}, \mathrm{V}) = C$ is conserved. Also define the more restricted coordinate set:

$$\mathbf{R}_i = R_i(t, \mathbf{r}_o, \mathbf{v}_o),\ \mathbf{V}_i = V_i(t, \mathbf{r}_o, \mathbf{v}_o);\ i = 1,2,3;\neq I \tag{5}$$

where I is an integer to be specified later.

3.2. POINCARÉ MA}' DEFINITION

Choose a plane of interest tangent to one of the fundamental coordinate planes, $\mathbf{r}_I$ = c, where I is the index mentioned previously. Assume that a particle started on this surface at time t = O will return to it at a future time t = T, and that the trajectory is transverse to this plane at both crossings (VI(0), $\mathbf{v}_I(T) \neq$ O). The time T is implicitly defined by the mapping between $\mathbf{r}_I(0)$ = c and $\mathbf{r}_I(T) = c$. Due to the transversal assumptions and the form of the Jacobi integral, VI may be solved for as a function of the Jacobi constant C. Performing these reductions produces a 4-dimension Poincaré map parameterized by the Jacobi constant C and the choice of Poincaré surface (c and I):

$$[\mathbf{R}_i(0; C, c), \mathbf{V}_i(0;\ \mathrm{C},\ \mathrm{c})] \rightarrow [\mathbf{R}_i(T; C, c), \mathbf{V}_i(T; C, c)] \tag{6}$$

3.3. MONODROMY MATRIX COMPUTATION

Now the 4 x 4 monodromy matrix associated with the above Poincaré map is derived. Recall the definition of the state transition matrix @(t):

$$\dot{\Phi}(t) = \frac{\partial(\dot{\mathbf{r}}(t), \dot{\mathbf{v}}(t))}{\partial(\mathbf{r}(t), \mathbf{v}(t))}\Phi(t);\ \Phi(0) = U \tag{7}$$

where U denotes the identity matrix. Φ is a 6 x 6 matrix and is most easily understood as the partial derivative of the state of the Lagrangian system at time t with respect to the state of the Lagrangian system at time t = O. q'bus,

$$\begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{bmatrix} = \Phi(T) \begin{bmatrix} \delta\mathbf{r}_o \\ \delta\mathbf{v}_o \end{bmatrix} \tag{8}$$

will be reduced to the 4 x 4 monodromy matrix of the Poincaré map in Equation 6, the form of this map will be

$$\begin{bmatrix} \delta\mathbf{R} \\ \delta\mathbf{V} \end{bmatrix} = A \begin{bmatrix} \delta\mathbf{R}_o \\ \delta\mathbf{V}_o \end{bmatrix} \tag{9}$$

3.3.1. *Reduction to the Poincaré Surface*

First specify $\mathbf{r}_I(0) = 0$, thus constraining the initial conditions to remain on the Poincaré surface. Next force the first return mapping to remain on the Poincaré surface. To do this a time variation in the map must be introduced, yielding:

$$\begin{bmatrix} \delta\mathbf{r}(T+\delta T) \\ \delta\mathbf{v}(T+\delta T) \end{bmatrix} = \Phi(T) \begin{bmatrix} \delta\mathbf{r}_o \\ \delta\mathbf{v}_o \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{r}}(T) \\ \dot{\mathbf{v}}(T) \end{bmatrix} \delta T \tag{10}$$

The condition to enforce is:

$$\Phi(T)_I \begin{bmatrix} \delta\mathbf{r}_o \\ \delta\mathbf{v}_o \end{bmatrix} + \dot{\mathbf{r}}_I(T)\delta T = 0 \tag{11}$$

where Φ_I denotes the Ith row and Φ^J denotes the Jth column of the matrix Φ. This equation may be solved for δT, due to the assumed transversality of the Poincaré map, yielding:

$$\begin{bmatrix} \delta\mathbf{R}(T+\delta T) \\ \delta\mathbf{v}(T+\delta T) \end{bmatrix} = \left(\Phi(T) - \frac{1}{\mathbf{v}_I(T)} \begin{bmatrix} \dot{\mathbf{r}}(T) \\ \dot{\mathbf{v}}(T) \end{bmatrix} \Phi(T)_I \right)^j_i \begin{bmatrix} \delta\mathbf{R}_o \\ \delta\mathbf{v}_o \end{bmatrix} \tag{12}$$
$$i,j = 1,\cdots,6; \neq I$$

a 5 x 5 map which takes the Poincaré surface to itself in a neighborhood of the nominal map,

3.3.2. Energy Reduction

To reduce Equation]2 to an energy conserving 4 x 4 map it must be constrained to lie on a single energy surface, corresponding to the condition:

$$0 = J_{\mathbf{R}}\delta\mathbf{R} + J_{\mathbf{V}}\delta\mathbf{V} + J_{\mathbf{v}_I}\delta\mathbf{v}_I \tag{13}$$

where J is the Jacobi integral and J_x is the partial of J with respect to X. Due to the transversality assumptions and the form of the Jacobi integral it is possible to solve for the variation $\delta\mathbf{v}_I$ at t = O which constrains the mapping to a single energy surface. Due to energy conservation the 5 variables at the first return to the surface will also conform to Equation 13, and thus $\delta\mathbf{v}_I(T)$ may be ignored as it can he reconstructed from the 4 other variables $\delta\mathbf{R}(T)$ and $\delta\mathbf{V}(T)$.

Substituting for $\delta\mathbf{v}_I(0)$ and reducing the indeces results in the final monodromy mapping:

$$\begin{bmatrix} \delta\mathbf{R} \\ \delta\mathbf{V} \end{bmatrix} = A \begin{bmatrix} \delta\mathbf{R}_o \\ \delta\mathbf{V}_o \end{bmatrix} \tag{14}$$

$$A = \left[\left(\Phi(T) - \frac{1}{\mathbf{v}_I(T)} \begin{bmatrix} \dot{\mathbf{r}}(T) \\ \dot{\mathbf{v}}(T) \end{bmatrix} \Phi(T)_I \right)^j_i - \frac{1}{\mathbf{v}_I(T)} \left(\Phi(T) - \frac{1}{\mathbf{v}_I(T)} \begin{bmatrix} \dot{\mathbf{r}}(T) \\ \dot{\mathbf{v}}(T) \end{bmatrix} \Phi(T)_I \right)^{I+3}_i [J_{\mathbf{R}}(0) J_{\mathbf{V}}(0)] \right] \tag{15}$$

$$i,j = 1,\cdots,6; \neq I, I+3$$

This map is energy preserving and carries a neighborhood of a Poincaré surface into itself. When close to periodic orbits, this mapping does not generically have unity eigenvalues and may be used to converge upon fixed points of the 4 dimensional Poincaré map.

3.4. RELATION TO A HAMILTONIAN SYSTEM

Applying the Legendre transformation to the original Lagrangian system, the matrix A will retain its same numerical value, as for any given Poincaré map the items within the matrix may be viewed as fixed parameters of the map. The variables $\delta\mathbf{R}$ and $\delta\mathbf{V}$ will transform linearly to the corresponding Hamiltonian variables as the Legendre transformation is linear in Cartesian coordinates, is invertible and depends only on parameters which are constant for the specific Poincaré map.

Thus Equation 15 is a similarity transformation away from a Hamiltonian system, and its eigenvalues may be interpreted as the eigenvalues of a Hamiltonian system. This frees one from computing explicit canonical transformations while allowing one to utilize the theoretical results from Hamiltonian systems.

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References

1. Chauvineau, B., Farinella, P. and Miguard, F. (1993) Planar Orbits about a Triaxial Body: Application to Asteroidal Satellites, *Icarus* 105, pp. 370- 384. '
2. MacMillan, W.D. (1936) Dynamics *of* Rigid Bodies, McGraw-Hill.
3. March at, C. (1990) *The Three-Body Problem*, Elsevier.
4. Scheeres, D. J., Ostro, S. J., Hudson, R. S., Werner, R.A. (1995) Orbits Close to Asteroid 4769 Castalia, submitted to *Icarus* April, 1995.
5. Scheeres, D. J., (1994) Dynamics About Uniformly Rotating Tri-axial Ellipsoids, *Icarus* 110, pp. 225–238.
6. Werner, R.A. (1994) The Gravitational Potential of a Homogeneous Polyhedron *Celestial* Mechanics 59, pp. 253--278.